Incomplete Markets and Heterogeneous Agents

The Aiyagari (1994) Model and Extensions

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The Representative-Agent Benchmark and Its Limits

Lucas(1987, 2003): Welfare Cost of Business Cycles

Lucas calculates welfare cost of business cycles as fraction of consumption HH would be willing to pay to eliminate BC fluctuations under complete markets

Result: A household would pay only $\approx 0.05\%$ of consumption to eliminate all aggregate volatility — trivially small.

Why might this be be too small? Two bundled assumptions:

1. Complete markets: idiosyncratic risk fully insured $\Rightarrow$ consumption/wealth distribution irrelevant
2. Representative agent: Single optimizing HH represents the aggregate

Aiyagari maintains ex-ante identical HH but drops complete markets — distribution becomes a state variable even without preference heterogeneity.

Today's question

What happens to aggregate outcomes and welfare when markets are *incomplete* and households face *uninsurable idiosyncratic risk*?

Roadmap: empirical motivation $\rightarrow$ Aiyagari model $\rightarrow$ quantitative results $\rightarrow$ extensions

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Empirical Evidence for Heterogeneity

Three facts that motivate moving beyond the RA model:

1. **Earnings losses in recessions are highly concentrated**
   Guvenen et al. (2014): the distribution of earnings growth is strongly left-skewed in downturns — large losses concentrated at the bottom, not shared uniformly. Confirmed and extended in Guvenen et al. (2021) using administrative data on millions of workers.
2. **Many households are "wealthy hand-to-mouth"**
   Kaplan et al. (2014): $\approx 1/3$ of U.S. households hold positive net worth but little *liquid* wealth — they have high MPCs despite non-trivial assets.
3. **Consumption responds strongly to transitory income**
   Parker et al. (2013): households spent 12–30% of 2008 stimulus payments on nondurables on impact (50–90% including durables) — far above what the permanent income hypothesis predicts.

Common thread

A non-trivial fraction of households faces binding or near-binding liquidity constraints. Aggregate dynamics depend on the *distribution* of wealth, not just its mean.

Select Macro Applications

Fiscal Multiplier Heterogeneity

(1) Aggregate MPC: constrained households have high MPC $\Rightarrow$ aggregate MPC exceeds RA prediction; fiscal multiplier larger.

(2) Targeting: distribution of MPC's across households means transfers to low-wealth HH generate larger stimulus than uniform lump sum — composition of fiscal policy matters, not just size.

Heterogeneous Agent New Keynesian: Monetary Transmission

In RA: transmission works exclusively through intertemporal substitution (Euler equation).

In HANK: *income channel* operates broadly — rate cuts raise labor demand, lifting wages and employment.

Connecting thread: All applications require the *joint distribution* of income and assets — exactly what Aiyagari delivers in the simplest credible framework.

Bewley-Huggett-Aiyagari

Neoclassical production economy with uninsurable idiosyncratic labor income risk and a borrowing constraint

The Individual Household Problem

Household i faces idiosyncratic labor income $y_t = w \cdot z_t$, where z_t follows an AR(1):

$$\log z_{t+1} = \rho_z \log z_t + \sigma_\varepsilon \varepsilon_{t+1}, \quad \varepsilon \sim \mathcal{N}(0,1)$$

Innovation std: $\sigma_\varepsilon = \sigma_z \sqrt{1-\rho_z^2}$. Unconditional std: $\sigma_z \equiv \sigma_\varepsilon / \sqrt{1-\rho_z^2}$. **Calibration targets σ_z (unconditional).**

Recursive problem:

$$V(a,z) = \max_{c \geq 0,\ a' \geq -\phi} \left\{ u(c) + \beta \mathbb{E}_z \left[V(a', z') \right] \right\}$$

subject to: $c + a' = (1+r)a + wz, \quad a' \geq -\phi$

Euler inequality (key object):

$$u'(c) \geq \beta(1+r)\,\mathbb{E}_z \left[u'(c') \right]$$

with equality iff the borrowing constraint $a' \geq -\phi$ is *not binding*.

Two forces on savings

- **Impatience:** $\beta(1+r) < 1$ pushes toward borrowing
- **Precautionary motive:** $u'' < 0, u''' > 0$ (CRRA) pushes toward saving as a buffer

Constraint matters even when slack

Even HH *not* at the constraint today save more because they *fear* hitting it tomorrow. *Analogy*: one does not have to crash to drive more carefully.

Firm Problem and Production Technology

Production technology: Cobb-Douglas with constant returns to scale:

$$Y = F(K, L) = AK^\alpha L^{1-\alpha}, \qquad \alpha \in (0, 1) \tag{1}$$
$$K_{t+1} = (1-\delta)K_t + I_t \tag{2}$$

Implies capital rental rate $r^k = r + \delta$

Normalize $L = 1$ (inelastic labor supply). All quantities are per unit of labor.

Competitive firm problem: firms rent capital at spot prices taking (r, w) as given:

$$\max_{K,L} \left\{ AK^\alpha L^{1-\alpha} - (r+\delta)K - wL \right\}$$

First-order conditions (with $L = 1$):

$$F_K = \alpha A K^{\alpha-1} = r + \delta \qquad\qquad F_L = (1-\alpha)AK^\alpha = w$$

Capital demand: $r^*(K)$

$$r^*(K) = \alpha A K^{\alpha-1} - \delta$$

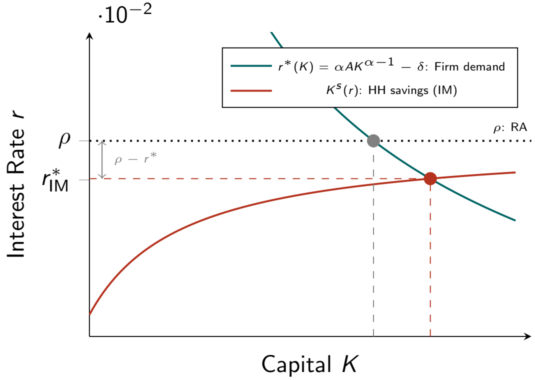
Strictly decreasing in K. Pins the firm side of the GE.

Wage schedule: $w(r)$

$$w(r) = (1-\alpha)A\left(\frac{\alpha A}{r+\delta}\right)^{\alpha/(1-\alpha)}$$

General Equilibrium: Why $r^* < \rho$

Household side: aggregate savings $K^s(r)$ increasing in r (standard supply) with horizontal asymptote at $r = \rho^*$



Key result: Despite individual uncertainty, the economy converges to a unique stationary distribution $\Phi^*(a, z)$ over asset-income pairs.

RA equilibrium: $r = \rho$ (Euler eq.), K^*_{RA} pinned by $r^*(K) = \rho$. No precautionary motive $\Rightarrow$ no wedge.

IM equilibrium: precautionary savings shifts K^s rightward $\Rightarrow K^*_{\text{IM}} > K^*_{\text{CM}}$, $r^*_{\text{IM}} < \rho$.

Liquidity premium:

$$\underbrace{\rho - r^*_{\text{IM}}}_{\text{liquidity premium}} > 0$$

- Premium rises with σ_z (more risk) and tightness of ϕ (tighter constraint)
- r^* can fall below zero even with $\rho > 0$

Calibration and Equilibrium

Parameter values (Aiyagari 1994):

Par.	Val.	Target / Source
β	0.96	Real interest rate $\approx 4\%$ (annual)
γ	2.0	EIS $= 1/\gamma \approx 0.5$
δ	0.08	Capital depreciation (annual)
α	0.36	Capital income share, U.S. data
ρ_z	0.9	Persistence: Floden and Lindé 2001
σ_z	0.2	Uncond. std: Aiyagari 1994 Table I
ϕ	0.0	No borrowing (baseline)

F&L innovation std $\hat{\sigma}_\varepsilon \approx 0.21$ implies unconditional std ≈ 0.52 — higher risk than here. Here $\sigma_\varepsilon = \sigma_z\sqrt{1-\rho_z^2} \approx 0.087$. Rouwenhorst (1995) preferred over Tauchen for high ρ_z.

Equilibrium algorithm: Bracket: $r \in (-\delta, \rho)$

1. Guess $r^{(0)} \Rightarrow$ compute w from firm FOC
2. Solve HH problem $\Rightarrow$ obtain policy $a'(a, z)$
3. Simulate $\Phi^*(a, z) \Rightarrow$ aggregate $K^s = \int a \, d\Phi^*$
4. **Bisection on** r**:** $K^d(r) = \left(\frac{\alpha A}{r+\delta}\right)^{1/(1-\alpha)}$.
   $K^s(r)\uparrow$, $K^d(r)\downarrow \Rightarrow$ unique crossing.
   $K^s > K^d$: lower $r_{\max}$; else raise $r_{\min}$.
5. Repeat until $|K^s - K^d| < \varepsilon$

Two HH solvers (see slide 11)

- VFI: general, slower
- EGM: exploits Euler equation, 3–5$\times$ faster

Julia Implementation: Code Architecture and the EGM

Standard VFI approach:

1. Fix grid of current assets $\{a_j\}$
2. For each (a_j, z_k): solve $\max_{a'} u(c) + \beta \mathbb{E} V(a', z')$ via numerical optimization
3. Iterate V until convergence

Bottleneck: nonlinear optimization at every grid point — severe curse of dimensionality

Endogenous Grid Method (Carroll 2006):

1. Initial guess of policy function $c_1(a, z)$
2. Fix grid of *end-of-period* assets $\{a'_j\}$
3. Invert the Euler equation analytically:

$$c = [\beta(1+r)\,\mathbb{E}u'(c')]^{-1/\sigma}$$

4. Recover current assets: $a = c + a' - wz$
5. Interpolate to get policy c_2 on fixed $\{a_j\}$ grid
6. Iterate until $||c_k - c_{k-1}|| < tol$

Code structure (`Aiyagari_functions.jl`):

- `rouwenhorst()`: AR(1) discretization
- `invariant_dist!`: Compute invariant distribution $\Phi(a, z)$
- `solve_model_egm()`: EGM solver (default)
- `general_equilibrium()`: GE loop with bisection on r
- `compute_mpc()`, `lorenz_curve()`, `compute_gini()`: distributional statistics
- `welfare_cost_idiosyncratic_risk()`: welfare calculations

EGM as type of Euler equation iteration

1. Uses optimality information from Euler equation
2. Avoids root finding at each state

Baseline Quantitative Results

Aiyagari (1994) Table I — Extended

Baseline: $\beta = 0.96$, $\gamma = 2$, $\alpha = 0.36$, $\delta = 0.08$, $b = 0$, $A = 1$

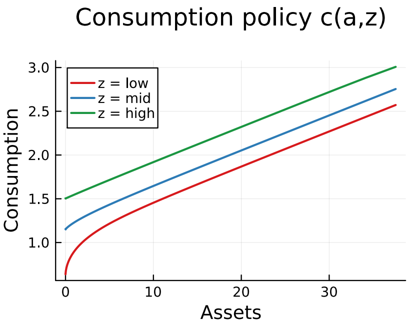
	$\sigma_z = 0.2$				$\sigma_z = 0.4$				CM
ρ_z	0.0	0.3	0.6	0.9	0.0	0.3	0.6	0.9	—
$r^*(\%)$	4.10	4.06	3.98	3.79	3.90	3.71	3.37	2.77	4.17
K^*	5.49	5.52	5.58	5.72	5.64	5.78	6.06	6.59	5.45
Liq. prem. (pp)	0.06	0.10	0.18	0.38	0.27	0.45	0.80	1.39	0.00
Agg. MPC	0.054	0.056	0.061	0.089	0.064	0.069	0.078	0.106	0.0417
% at constr.	0.19	0.23	0.49	2.91	0.18	0.28	0.59	3.46	0.00

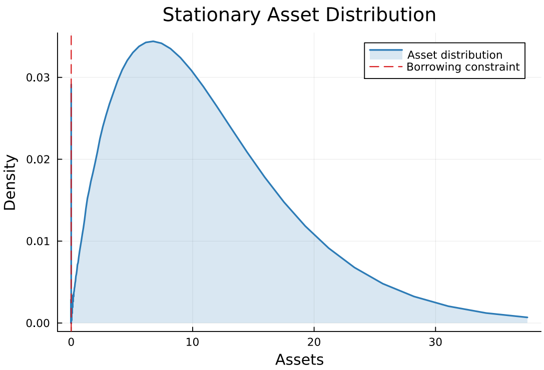
σ_z irrelevant for CM: $r^* = \rho$, MPC$\approx r^*$. Note: conditional standard deviation $\sigma_\varepsilon = \sigma_z\sqrt{1-\rho_z^2}$ falls with persistence.

Three key patterns (preview)

1. Incomplete markets $\Rightarrow$ higher K^*, lower r^* (precautionary savings)
2. Liquidity premium rises with income volatility σ_z
3. Aggregate MPC far exceeds the RA benchmark — matters for fiscal policy

Consumption Policy and Asset Distribution





Key features:

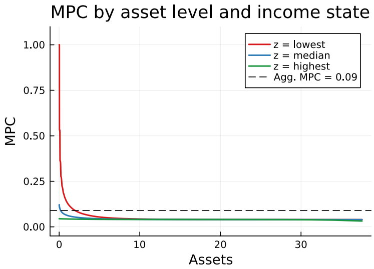
- **Concavity** of $c(a,z)$: MPC declining in wealth — poor HH consume more out of an extra dollar
- Kink at $a = -\phi$: constraint binds $\Rightarrow$ $c = (1+r)a + wz + \phi$

Key features:

- **Right skew**: mass concentrated at low asset levels
- **Point mass at** $-\phi$: constrained households
- Long right tail driven by high-z HH who accumulate

MPC Distribution, Inequality, and Welfare

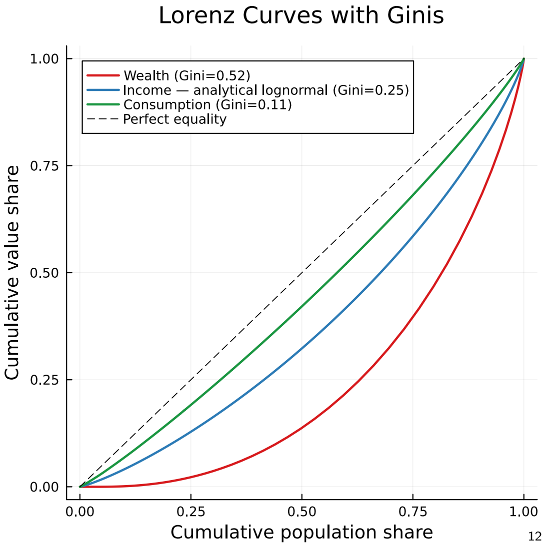
Cross-sectional MPC distribution:



- Aggregate MPC above RA benchmark
- Wide dispersion: constrained HH have MPC ≈ 1; wealthy HH MPC $r^* \approx 0.042$ (CM benchmark)

Percentile	Wealth share (%)
Top 1%	4.9%
Top 10%	31.9%
Top 50%	85.5%

Lorenz curves and Gini coefficients:



Interpretation of Inequality Measures

- Ranking: $\text{Gini}_{\text{wealth}} > \text{Gini}_{\text{income}} > \text{Gini}_{\text{consumption}}$
- Intuition: self-insurance via savings amplifies wealth inequality relative to income; consumption smoothing mitigates consumption inequality relative to income
- Aiyagari under-generates top-end wealth concentration relative to U.S. data $\Rightarrow$ motivates extensions: entrepreneurs (superstar firms, network effects), stochastic returns, learning/career paths, rent seeking

Welfare: What Aiyagari Can and Cannot Say

Willingness to pay to eliminate idiosyncratic risk

Find $\lambda(a,z)$: fraction of CM consumption foregone to move from IM to CM:

$$V^{\text{IM}}(a,z) = \frac{u((1-\lambda)C^*)}{1-\beta}, \quad C^* \equiv \int c \, d\Phi^*$$

Solving for CRRA:

$$\lambda(a,z) = 1 - \frac{\left[(1-\gamma)(1-\beta)V^{\text{IM}}(a,z)\right]^{1/(1-\gamma)}}{C^*}$$

Results ($\rho_z = 0.9$, $\sigma_z = 0.2$, $\gamma = 2$):

- Aggregate $\bar{\lambda} = 1.46\%$ — **29× Lucas (1987)'s 0.05%**
- Bottom decile: $\lambda = 21.5\%$ — 14.7× the aggregate

CM benchmark is not a Pareto improvement

C^* is mean consumption — below wealthy HH's IM consumption. Wealthy HH have $\lambda < 0$: they prefer IM. $\bar{\lambda} > 0$ under utilitarian weighting: poor HH gains dominate.

GE caveat

CM benchmark uses C^*_{IM} — mean consumption at the IM capital stock. True CM equilibrium has different K^* (no precautionary motive $\Rightarrow K^*_{\text{CM}} < K^*_{\text{IM}}$) and hence different C^*_{CM}.

What Aiyagari cannot say

- Welfare cost of aggregate fluctuations per se
- Transition dynamics (stationary dist. only)

Toward aggregate shocks: Krusell & Smith (1998)

Extension: $\tilde{z}_t = z_t \cdot Z_t$.

Storesletten et al. (2004): idiosyncratic earnings risk is strongly countercyclical — conditional std rises 75% from peak to trough (PSID GMM).

Storesletten, Telmer & Yaron (2001): incorporating this countercyclical risk into an OLG IM model raises welfare cost of BC an order of magnitude above Lucas (1987).

Extension I: Labor Markets and the Job Ladder

Reinterpret idiosyncratic risk as *employment status* $e_i \in \{0, 1\}$ and productivity rank on a job ladder.

Key mechanisms:

- **Unemployment spells = forced decumulation:** an unemployed HH draws on precautionary savings — can hit the constraint if the spell is long enough
- **Job ladder dynamics:** employed HH climb productivity rank over time; displacement destroys rank and induces decumulation

Aggregate MPC and Policy Implications

- Recession: Job finding rate $f(\theta) \downarrow$ — employed HH raise precautionary savings (MPC $\downarrow$); unemployed deplete assets faster (MPC $\uparrow$), share u rises
- **Net effect:** unemployed channel dominates empirically Kaplan et al. (2018) — aggregate MPC *rises* in recessions
- **Implication:** fiscal multipliers are largest precisely when unemployment is high
- **Requires knowing $\Phi^*(a, e)$:** a wealthy recently-displaced worker has low MPC; asset-poor long-term unemployed has MPC ≈ 1. Employment status alone is insufficient.

Extension II: Liquidity Constraints and Two Assets (I) — Environment

Assets (all quantities in units of the consumption good):

- **Real balances** $m_t \equiv M_t/P_t \geq 0$: zero nominal return; real return $-\pi$ (inflation erodes value). Perfectly liquid.
- **Capital** $k \geq 0$: net return $r_k = F_K - \delta$; fraction $\eta \in (0,1)$ pledgeable as collateral to competitive financiers — illiquid at the margin

Dual role of capital

Capital serves as productive asset *and* collateral

Budget and liquidity constraints:

$$c_t + (1+\pi)m_{t+1} + k_{t+1} = wz_t + m_t + (1+r_k)k_t + T$$
$$c_t \leq \underbrace{wz_t}_{\text{current income}} + \underbrace{m_t}_{\text{liquid savings}} + \underbrace{\eta k_t}_{\text{collateral loan}}$$

where $T = \pi m^*$ is lump-sum seigniorage rebate

Timing within period t:

1. Enter with (m_t, k_t) — chosen before z_t known
2. Idiosyncratic shock z_t realized; labor income wz_t received
3. Consume c_t subject to liquidity constraint; repay collateral loan ηk_t
4. Choose new portfolio (m_{t+1}, k_{t+1}) before z_{t+1} realized

Extension II: Optimality and Inflation Channels (II)

General solution for consumption:

$$c \;=\; \min\{c^*,\; wz + m + \eta k\}$$

Unconstrained HH choose c^* (standard Euler); constrained HH exhaust available liquidity. Constraint binds iff $c^* > wz + m + \eta k$.

Interior conditions

Capital ($k' > 0$): always held in GE — Inada condition ($F_K \to \infty$ as $K \to 0$) guarantees positive demand.

Money ($m' > 0$): holds only if $\pi < \bar{\pi}(\eta)$. Above threshold the economy goes cashless: $m' = 0$ and the money Euler holds as a weak inequality. Higher η raises $\bar{\pi}$ — better collateral reduces the insurance value of money.

Inflation channels

- **Real balances effect:** inflation disincentivizes money holdings
- **Tobin substitution effect:** inflation induces substitution into capital
- **Distributional effect:** inflation tax falls disproportionately on high-m households, but lump-sum transfer $T = \pi m^*$ is uniform — redistributes toward poorer households

Extension II: Liquidity Premia and Comparative Statics (III)

Key aggregation result (averaging individual Eulers over Φ^*, using stationarity and law of iterated expectations):

$$\frac{\bar{\mu}}{\bar{\lambda}} \approx i \equiv \rho + \pi \qquad \Longrightarrow \qquad \boxed{F_K^* = \rho + \delta - i\eta}$$

Liquidity premia on money and capital

$$\rho - (-\pi) = \rho + \pi = i \qquad \text{(money)}$$
$$\rho - r_k^* = i\eta \qquad \text{(capital)}$$

Comparative statics:

1. $\uparrow \eta$ or $\uparrow \pi$: compress F_K^*, raise K^*
2. Friedman rule $i = 0$: $F_K^* = \rho + \delta$ — Ramsey allocation restored

Financial shock $(\eta \downarrow)$ — amplification:

1. Collateral $\eta k \downarrow$ tightens liquidity constraint
2. $\mu \uparrow$ — constrained HH cut c

Limit $\sigma_z \to 0$:

1. $\Phi^*(m, k, z)$ collapses to a point — precautionary motive vanishes
2. K^*, M^* fall to CM levels; excess precautionary accumulation disappears
3. Inflation welfare cost becomes uniform across HH
4. Liquidity premium $i\eta$ survives — driven by η and π, not by idiosyncratic risk

Appendix

Extension II: Household Problem and Euler Equations (II)

Recursive problem: $V(m,k,z) = \max_{c,\, m',\, k' \geq 0} \{u(c) + \beta\, \mathbb{E}_z[V(m',k',z')]\}$

Constraints: Budget: $c + (1+\pi)m' + k' = wz + m + (1+r_k)k + T$ Liquidity: $c \leq wz + m + \eta k$

Lagrangian: multipliers $\lambda \geq 0$ (budget), $\mu \geq 0$ (liquidity). FOC$_c$: $\lambda = u'(c) - \mu$.

Euler equations (interior $m' > 0$, $k' > 0$; envelope: $V_m = \lambda + \mu$, $V_k = \lambda(1+r_k) + \mu\eta$):

$$\text{(money):} \quad \lambda_t(1+\pi) = \beta\, \mathbb{E}_t[\lambda_{t+1} + \mu_{t+1}] \tag{3}$$

$$\text{(capital):} \quad \lambda_t = \beta\, \mathbb{E}_t[\lambda_{t+1}(1 + r_{k,t+1}) + \mu_{t+1}\, \eta] \tag{4}$$

If liquidity constraint does not bind ($\mu_t = 0$), then Eulers satisfy

$$u'(c_t)(1+\pi) = \beta \mathbb{E}_t(\lambda_{t+1} + \mu_{t+1}) \tag{5}$$

$$u'(c_t) = \beta \mathbb{E}_t[\lambda_{t+1}(1+r_k) + \mu_{t+1}\eta] \tag{6}$$

Let c_t^* be solution for unconstrained problem
General solution for c:

$$c = \min\{c^*, wz + m + \eta k\} \tag{7}$$

Conditions for interior solution and inflation channels

Interior conditions and portfolio indeterminacy

Interior $k' > 0$**:** guaranteed in GE — Inada ($F_K \to \infty$ as $K \to 0$).
Interior $m' > 0$**:** requires $\pi < \bar{\pi}(\eta)$. Above threshold, $m' = 0$ and (3) holds as inequality.

Inflation channels

- Real balances effect: inflation disincentivizes money holdings
- Tobin substitution effect: inflation induces substitution into capital
- Distributional effect: inflation tax falls disproportionately on high$-m$ HH but lump-sum transfer $T = \pi m^*$ is uniform–redistributes to poorer HH

Extension II: Aggregation Under Stationarity (III)

Define $\bar{\lambda} \equiv \mathbb{E}_{\Phi^*}[\lambda]$, $\bar{\mu} \equiv \mathbb{E}_{\Phi^*}[\mu]$ (time-invariant by stationarity). r_k^* aggregate $\Rightarrow$ factors out; law of iterated expectations: $\mathbb{E}_{\Phi^*}[\mathbb{E}_t[\lambda_{t+1}]] = \bar{\lambda}$.

Note: $\bar{\mu} > 0$ requires a positive-measure set of constrained households in Φ^*

Aggregated Euler equations:

$$\text{(money)}: \quad \bar{\lambda}(1+\pi) = \beta(\bar{\lambda} + \bar{\mu}) \tag{8}$$

$$\text{(capital)}: \quad \bar{\lambda} = \beta\bar{\lambda}(1 + r_k^*) + \beta\bar{\mu}\,\eta \tag{9}$$

Note that $\frac{\bar{\mu}}{\bar{\lambda}}$ represents economy-wide average liquidity wedge

From (8):

$$\frac{\bar{\mu}}{\bar{\lambda}} = \frac{1+\pi}{\beta} - 1 = (1+\pi)(1+\rho) - 1 \approx \boxed{i \equiv \rho + \pi}$$

Caveat: Valid when $m' > 0$ for a positive-measure set of households. At high π, $m' = 0$.

Substituting into (9):

$$1 = \beta(1 + r_k^*) + \beta\, i\, \eta \implies \boxed{F_K^* = \rho + \delta - i\eta}$$

Liquidity premium of an asset is measured as gap between ρ and asset return

Liquidity premia on money and capital

$$\rho - (-\pi) = \rho + \pi = i$$
$$\rho - r_k^* = i\eta$$

Extension II: Implications and Comparative Statics (IV)

Effect of η and i on equilibrium:

$$F_K^* = \rho + \delta - i\eta, \qquad i \equiv \rho + \pi$$

1. $\uparrow \eta$ or $\uparrow \pi$: both compress F_K^*, raise K^*.
2. At $i = 0$ (Friedman rule) and $\eta = 0$: standard Ramsey $F_K^* = \rho + \delta$.

Financial shock ($\eta \downarrow$) — amplification:

1. Collateral value $\eta k \downarrow$ tightens liquidity constraint
2. $\mu \uparrow$ — constrained HH cut c
3. Aggregate demand falls $\Rightarrow q_k \downarrow$ (if asset prices endogenous via convex adjustment costs)
4. Lower q_k further tightens collateral: Kiyotaki-Moore spiral

Low-m HH (high μ) cut consumption most — amplification rises with dispersion in Φ^*.

Role of $\sigma_z \to 0$:

1. Stationary dist. $\Phi^*(m, k, z)$ collapses to point: precautionary motive vanishes
2. K^* and M^* fall to representative-agent levels — excess precautionary accumulation disappears, but monetary premium $i\eta$ remains
3. Inflation welfare cost uniform across HH
4. But liquidity premium unaffected (depends on i and η)

Appendix: Equilibrium Algorithm — Fixed-Point Structure

Key simplification: K^* is pinned *analytically* by the aggregated capital Euler (at interior money solution):

$$K^* = F_K^{-1}(\rho + \delta - i\eta + \delta), \qquad i \equiv \rho + \pi$$

The only genuine fixed-point iteration is over aggregate real balances m^*. Algorithm:

Outer loop — iterate on m^*:

1. **Guess** $m^{*(0)}$. Compute factor prices $r_k = F_K(K^*, 1) - \delta$, $w = F_L(K^*, 1)$, transfer $T = \pi m^*$.
2. **Solve household problem** on sparse grid (Slide A2) using EGM (Slide A3). Obtain policy functions $c(m, k, z)$, $m'(m, k, z)$, $k'(m, k, z)$.
3. **Simulate** stationary distribution $\Phi^*(m, k, z)$ via Young (2010) histogram method on Sobol nodes.
4. **Update:**

$$m^*_{\text{new}} = \int m' \, d\Phi^*, \qquad K^*_{\text{new}} = \int k' \, d\Phi^*$$

Repeat until $|m^*_{\text{new}} - m^*| < \varepsilon$.

K^* consistency check

The analytically pinned K^* and the simulated $K^*_{\text{new}} = \int k' \, d\Phi^*$ must agree in equilibrium. A discrepancy signals either a numerical error or that the interior money condition is violated for a positive measure of households — in which case the Fisher equation no longer pins $\bar{\mu}/\bar{\lambda} = i$ and K^* must be found from capital market clearing directly.

The check is therefore a diagnostic for the validity of the interior solution assumption.

Appendix: Sparse Grid and Chebyshev Approximation

Problem: state space is (m, k, z) — two continuous asset dimensions plus N_z discrete income states. A naïve tensor grid with $N_m \times N_k$ points per z-state scales poorly.

Sobol sequences (quasi-random grid)

Low-discrepancy sequences fill the $(m, k) \in [0, \bar{m}] \times [0, \bar{k}]$ space more uniformly than either regular tensor grids or pseudo-random draws.

Approximation error: $O\big((\log N)^d / N\big)$ vs. $O(N^{-p/d})$ for tensor grids — the curse of dimensionality is largely avoided for $d = 2$.

In practice: $N \approx$ 500–1000 Sobol nodes per z-state suffices, versus $200 \times 200 = 40{,}000$ for a comparable tensor grid.

Chebyshev polynomial approximation

Approximate the value function as:

$$V(m, k, z_j) \approx \sum_{p=0}^{P} \sum_{q=0}^{Q} \theta^j_{pq}\, T_p(\tilde{m})\, T_q(\tilde{k})$$

where $\tilde{m}, \tilde{k} \in [-1, 1]$ are affinely mapped asset values and T_p is the Chebyshev polynomial of degree p.

Coefficients θ^j_{pq} are updated each VFI iteration. Chebyshev nodes cluster near asset boundaries — where the liquidity constraint binds and policy functions have highest curvature — making approximation most accurate where it matters most.

Appendix: Endogenous Grid Method — Two-Region Policy Function

The household's optimality conditions split the state space into two regions according to whether the liquidity constraint binds.

Region 1 — Unconstrained ($\mu = 0$)

$\lambda = u'(c)$. Two Euler equations hold as equalities:

$$u'(c_t)(1+\pi) = \beta\,\mathbb{E}_t[u'(c_{t+1})]$$
$$u'(c_t) = \beta\,\mathbb{E}_t[u'(c_{t+1})(1+r_k)]$$

EGM inverts each analytically for c given tomorrow's policy, then recovers today's (m, k) from the budget constraint. No root-finding required.

Portfolio indeterminacy: with $\mu_{t+1} = 0$ tomorrow, the split between m' and k' is indeterminate unless the constraint binds with positive probability going forward. Money is valued only if $\Pr(\mu_{t+1} > 0) > 0$ — a theoretical requirement verified ex post by the solution.

Region 2 — Constrained ($\mu > 0$)

Consumption is pinned by the constraint:

$$c = wz + m + \eta k$$

Given c, $\lambda = u'(c) - \mu$ and the budget constraint pins the total portfolio outflow $m'(1+\pi) + k'$. The split between m' and k' within Region 2 is determined by the relative shadow values — both Eulers hold as inequalities and the binding one determines the corner. In practice: constrained households hold the minimum liquid portfolio consistent with next-period coverage.

Constraint locus and iteration

The boundary between regions is the locus $\{(m, k, z) : wz + m + \eta k = c^*(m, k, z)\}$ where c^* is the unconstrained optimal consumption. This boundary is computed endogenously each iteration on the Sobol grid. Chebyshev interpolation approximates it smoothly, avoiding the grid-point aliasing that would arise on a coarse tensor grid.

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